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# $L^1$-CONVERGENCE OF XHEVAT MODIFIED SUMS IN TWO DIMENSIONS

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**Abstract:** In this paper, X.Z. Krasniqi modified trigonometric sine sum is extended from one dimension to two dimension and the integrability and $L^1$ - convergence of double cosine series under new coefficients of numerical sequences has been studied.

**Keywords and Phrases:** Dirichlet kernel, Fejer kernel, Integrability, $L^1$- convergence, double trigonometric series.

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## 1. Introduction

Let

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\lambda_j\lambda_k q_{jk}^d \cos jx_1 \cos kx_2 \tag{1.1}$$

be the double cosine series on the positive quadrant $Q = [0,\pi]\times[0,\pi]$ of the two dimensional torus, where $\lambda_0 = 1/2$ and $\lambda_j = 1$ if $j \geq 1$; $q_{jk}^d$ are real coefficients and let

$$S_{mn}^d(x_1,x_2) = \sum_{j=0}^{m}\sum_{k=0}^{n}\lambda_j\lambda_k q_{jk}^d \cos jx_1 \cos kx_2 \qquad (m,n\geq 0)$$

be the rectangular partial sum of the series (1.1) and $f^d(x_1,x_2) = \lim_{m,n\to\infty} S_{mn}^d(x_1,x_2)$.

**Bounded Variation.** The real coefficients $\{q_{jk}^d\}$ form a null sequence of bounded variation if

$$q^d_{jk} \to 0 \quad as \quad \max(j,k) \to \infty, \tag{1.2}$$

and

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\Delta_{11} q^d_{jk}| < \infty, \tag{1.3}$$

where $\Delta_{11} q^d_{jk} = q^d_{jk} - q^d_{j+1,k} - q^d_{j,k+1} + q^d_{j+1,k+1}$.
The differences $\Delta_{ab} q^d_{jk}$ are defined as:

$$\Delta_{ab} q^d_{jk} = \sum_{i=0}^{a}\sum_{l=0}^{b} (-1)^{i+l} \binom{a}{i}\binom{b}{l} q^d_{j+i,k+l} \quad (j,k > 0).$$

for all pairs of non-negative integers $a$ and $b$.
Hence, the following recurrence relation holds:

$$\begin{aligned} \Delta_{00} q^d_{jk} &= q^d_{jk}, \\ \Delta_{ab} q^d_{jk} &= \Delta_{a-1,b} q^d_{jk} - \Delta_{a-1,b} q^d_{j+1,k} \quad (a \geq 1), \\ \Delta_{ab} q^d_{jk} &= \Delta_{a,b-1} q^d_{jk} - \Delta_{a,b-1} q^d_{j,k+1} \quad (b \geq 1). \end{aligned}$$

In 1991, Móricz [7] considered yet another condition which is stronger than the condition (1.3) namely

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\Delta_{11} q^d_{jk}| \ln(j+2)\ln(k+2) < \infty, \tag{1.4}$$

and proved the following result:

**Theorem 1.1.** *[7] If a double sequence $\{q^d_{jk}\}$ satisfies (1.2) and (1.4), then the sum $f^d$ of the series (1.1) is integrable, (1.1) is the Fourier series of $f^d$, and $||S^d_{mn} - f^d|| \to 0$ as $\min(m,n) \to \infty$.*

**Quasi Convex.** [7] A double sequence $\{q^d_{jk}\}$ is said to be *quasiconvex* if

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (j+1)(k+1)|\Delta_{22} q^d_{jk}| < \infty. \tag{1.5}$$

In this paper, we define a new class $\mathcal{S}_{jk}$ of coefficient sequences as follows:
**Class $\mathcal{S}_{jk}$.** A double sequence $\{q^d_{jk}\}$ is said to belong to class $\mathcal{S}_{jk}$ if (1.2) is satisfied and

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (j+1)(k+1)|\Delta_{22} q^d_{jk}| \ln(j+2)\ln(k+2) < \infty. \tag{1.6}$$

Clearly, the condition (1.6) is stronger than the condition (1.5).
Choose $\Delta_{22}q^d_{jk} = \dfrac{1}{j^4k^4}$. Clearly, $\{q^d_{jk}\}$ belongs to class $\mathcal{S}_{jk}$.

*"During literature survey, it is found that various authors such as Móricz ([4], [6], [7]), Kaur, Bhatia and Ram [2], Kaur and Bhatia [1] studied the integrability and $L^1$-convergence of double trigonometric series under different conditions on coefficient sequences and by considering various double modified trigonometric sums. These modified sums approximate their limit better than the classical double trigonometric series as they converge in $L^1$-metric to the sum of the trigonometric series whereas the classical series itself may not."*

The aim of this paper is to study the $L^1$-convergence of double trigonometric cosine series under the class $\mathcal{S}_{jk}$ by using a new modified double trigonometric sine sum (an extension of Xhevat [3] modified sine sums) defined as

$$f^d_{mn}(x_1, x_2) = \frac{1}{4\sin x_1 \sin x_2}\sum_{j=1}^{m}\sum_{k=1}^{n}\times$$

$$\left\{\sum_{r=j}^{m}\sum_{l=k}^{n}\Delta_{11}\left[\begin{array}{r}(q^d_{r-1,l-1} - q^d_{r+1,l-1} - q^d_{r-1,l+1} + q^d_{r+1,l+1})\\ \sin rx_1 \sin lx_2\end{array}\right]\right\}$$

(where $q^d_{j,k} = 0$ for either $j = 0$ or $k = 0$)

## 2. Lemma

The proof of our result is based upon the following lemma.

**Lemma 2.1.** *If $\{q^d_{jk}\}$ belongs to class $\mathcal{S}_{jk}$, then the following holds:*

$$(i)\quad \sum_{j=0}^{m}(j+1)|\Delta_{20}q^d_{jn}|\ln(j+2)\ln n = o(1),\ \text{as } n\to\infty.$$

$$(ii)\quad \sum_{k=0}^{n}(k+1)|\Delta_{02}q^d_{mk}|\ln(k+2)\ln m = o(1),\ \text{as } m\to\infty.$$

$$(iii)\quad m^a n^b \Delta_{ab}q^d_{mn}\ln m\ln n = o(1)\ \text{as } \min(m,n)\to\infty,\ \text{where } a,b\in\{0,1\}.$$

**Proof.** (i) Consider

$$\sum_{j=0}^{m}(j+1)|\Delta_{20}q^d_{jn}|\ln(j+2)\ln n \;=\; \ln n\sum_{j=0}^{m}(j+1)\sum_{k=n}^{\infty}|\Delta_{21}q^d_{jk}|\ln(j+2),$$

on applying summation by parts on right side of above equality, we get

$$\ln n \sum_{j=0}^{m}\sum_{k=n}^{\infty} |\Delta_{22} q^d_{jk}|(k+1)(j+1)\ln(j+2)$$

$$+\ln n \sum_{j=0}^{m} |\Delta_{21} q^d_{jn}| n(j+1)\ln(j+2)$$

$$\leq \sum_{j=0}^{m}\sum_{k=n}^{\infty} |\Delta_{22} q^d_{jk}|(j+1)(k+1)\ln(j+2)\ln(k+2)$$

$$+n\ln n \sum_{j=0}^{m}\sum_{k=n}^{\infty} |\Delta_{22} q^d_{jk}|(j+1)\ln(j+2)$$

$$\leq \sum_{j=0}^{m}\sum_{k=n}^{\infty} (j+1)(k+1)|\Delta_{22} q^d_{jk}|\ln(j+2)\ln(k+2)$$

$$+\sum_{j=0}^{m}\sum_{k=n}^{\infty} (j+1)(k+1)|\Delta_{22} q^d_{jk}|\ln(j+2)\ln(k+2).$$

Both the series on right hand side of above inequality are tending to zero by using the condition (1.6).

Hence, $\sum_{j=0}^{m}(j+1)|\Delta_{20} q^d_{jn}|\ln(j+2)\ln n = o(1),$ as $n\to\infty$, independent of $m$.

(ii) Similarly, $\sum_{k=0}^{n}(k+1)|\Delta_{02} q^d_{mk}|\ln(k+2)\ln m = o(1),$ as $m\to\infty$, independent of $n$.

The proof is on the same lines as that of the proof of (i).

(iii) **Case I.** When $a=1,\ b=1$

Consider

$$\begin{aligned}
\Delta_{11} q^d_{mn}\ln m\ln n &= \ln m\ln n \sum_{j=m}^{\infty}\Delta_{21} q^d_{jn} \\
&= \ln m\ln n \sum_{j=m}^{\infty}\sum_{k=n}^{\infty}\Delta_{22} q^d_{jk} \\
&\leq \frac{\ln m\ln n}{mn}\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}(j+1)(k+1)|\Delta_{22} q^d_{jk}|
\end{aligned}$$

$$mn|\Delta_{11}q^d_{mn}|\ln m\ln n \leq \sum_{j=m}^{\infty}\sum_{k=n}^{\infty}(j+1)(k+1)|\Delta_{22}q^d_{jk}|\ln(j+2)\ln(k+2)$$
$$= o(1),\ \ \min(m,n)\to\infty.\ \ \text{(by the given hypothesis)}$$

**Case II.** When $a=1,\ b=0$ or $a=0,\ b=1$
Consider

$$\begin{aligned}
\Delta_{10}q^d_{mn}\ln m\ln n &= \ln m\ln n\sum_{j=m}^{\infty}\Delta_{20}q^d_{jn}\\
&= \frac{\ln m\ln n}{m}\sum_{j=m}^{\infty}(j+1)|\Delta_{20}q^d_{jn}|\\
m\Delta_{10}q^d_{mn}\ln m\ln n &\leq \sum_{j=m}^{\infty}(j+1)|\Delta_{20}q^d_{jn}|\ln(j+2)\ln n = o(1),\ \ \min(m,n)\to\infty.
\end{aligned}$$

Similarly, $n\Delta_{01}q^d_{mn}\ln m\ln n = o(1)$ as $\min(m,n)\to\infty$.

**Case III.** When $a=0,\ b=0$
Consider

$$\begin{aligned}
q^d_{mn}\ln m\ln n &= \ln m\ln n\sum_{j=m}^{\infty}\Delta_{10}q^d_{jn}\\
&\leq \ln m\ln n\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}\Delta_{11}q^d_{jk},
\end{aligned}$$

performing summation by parts, we have

$$\begin{aligned}
q^d_{mn}\ln m\ln n &\leq \ln m\ln n\left[\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}k|\Delta_{12}q^d_{jk}| + \sum_{j=m}^{\infty}|\Delta_{11}q^d_{jn}|n\right]\\
&= \ln m\ln n\left[\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}k|\Delta_{12}q^d_{jk}| + n\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}|\Delta_{12}q^d_{jk}|\right].
\end{aligned}$$

Applying summation by parts on both the series of above equation, we get

$$q^d_{mn}\ln m\ln n = \ln m\ln n\left[\sum_{k=n}^{\infty}\left|\sum_{j=m}^{\infty}\Delta_{22}q^d_{jk}jk - \Delta_{12}q^d_{mk}mk\right|\right.$$

$$+ n\sum_{k=n}^{\infty}\left|\sum_{j=m}^{\infty}\Delta_{22}q_{jk}^{d}j - \Delta_{12}q_{mk}^{d}m\right|\Bigg]$$

$$\leq \ln m \ln n\left[\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}(j+1)(k+1)|\Delta_{22}q_{jk}^{d}| + \sum_{j=m}^{\infty}\sum_{k=n}^{\infty}(j+1)(k+1)|\Delta_{22}q_{jk}^{d}|\right.$$

$$\left.+\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}(j+1)(k+1)|\Delta_{22}q_{jk}^{d}| + \sum_{j=m}^{\infty}\sum_{k=n}^{\infty}(j+1)(k+1)|\Delta_{22}q_{jk}^{d}|\right]$$

$$\leq 4\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}(j+1)(k+1)|\Delta_{22}q_{jk}^{d}|\ln(j+2)\ln(k+2)$$

$$= o(1) \text{ as } \min(m,n)\to\infty.$$

Combining all the cases yields (iii).

## 3. Main Result

The main result of this chapter read as:

**Theorem 3.1.** *If a double sequence $\{q_{jk}^{d}\}$ belongs to the class $\mathcal{S}_{jk}$, then for $0 < x_1, x_2 < \pi$, the following holds true:*

(i) $\lim_{m,n\to\infty} f_{mn}^{d}(x_1,x_2) = f^{d}(x_1,x_2)$ *exists,*

(ii) $f^{d}(x_1,x_2) \in L^{1}(Q)$,

(iii) $||f_{mn}^{d} - f^{d}|| \to 0$ *as* $\min(m,n)\to\infty$.

**Proof.** Firstly consider,

$$f_{mn}^{d}(x_1,x_2) = \frac{1}{4\sin x_1 \sin x_2}\sum_{j=1}^{m}\sum_{k=1}^{n}$$

$$\left\{\sum_{r=j}^{m}\sum_{l=k}^{n}\Delta_{11}\left[\begin{matrix}(q_{r-1,l-1}^{d} - q_{r+1,l-1}^{d} - q_{r-1,l+1}^{d} + q_{r+1,l+1}^{d}) \\ \sin rx_1 \sin lx_2\end{matrix}\right]\right\}$$

$$
\begin{aligned}
f_{mn}^{d} &= \frac{1}{4\sin x_1 \sin x_2} \times \\
&\left\{ \sum_{j=1}^{m}\sum_{k=1}^{n}(q_{j-1,k-1}^{d} - q_{j+1,k-1}^{d} - q_{j-1,k+1}^{d} + q_{j+1,k+1}^{d}) \sin jx_1 \sin kx_2 \right. \\
&- \sum_{j=1}^{m}\sum_{k=1}^{n}(q_{j-1,n}^{d} - q_{j+1,n}^{d} - q_{j-1,n+2}^{d} + q_{j+1,n+2}^{d}) \sin jx_1 \sin(n+1)x_2 \\
&- \sum_{j=1}^{m}\sum_{k=1}^{n}(q_{m,k-1}^{d} - q_{m+2,k-1}^{d} - q_{m,k+1}^{d} + q_{m+2,k+1}^{d}) \sin(m+1)x_1 \sin kx_2 \\
&\left. + \sum_{j=1}^{m}\sum_{k=1}^{n}(q_{m,n}^{d} - q_{m+2,n}^{d} - q_{m,n+2}^{d} + q_{m+2,n+2}^{d}) \sin(m+1)x_1 \sin(n+1)x_2 \right\} \\
&= I_1 - I_2 - I_3 + I_4.
\end{aligned}
$$

Consider,

$$
I_1 = \frac{1}{4\sin x_1 \sin x_2} \sum_{j=1}^{m}\sum_{k=1}^{n}(q_{j-1,k-1}^{d} - q_{j+1,k-1}^{d} - q_{j-1,k+1}^{d} + q_{j+1,k+1}^{d}) \sin jx_1 \sin kx_2
$$

$$
\begin{aligned}
&= \frac{1}{4\sin x_1 \sin x_2} \left\{ \sum_{j=1}^{m}\sum_{k=1}^{n}(\sin(k+1)x_2 - \sin(k-1)x_2) q_{j-1,k}^{d} \sin jx_1 \right. \\
&\qquad - \sum_{j=1}^{m}\sum_{k=1}^{n}(\sin(k+1)x_2 - \sin(k-1)x_2) q_{j+1,k}^{d} \sin jx_1 \\
&\qquad - \sum_{j=1}^{m}\left(q_{j-1,n}^{d} - q_{j+1,n}^{d}\right) \sin jx_1 \sin(n+1)x_2 \\
&\qquad \left. - \sum_{j=1}^{m}\left(q_{j-1,n+1}^{d} - q_{j+1,n+1}^{d}\right) \sin jx_1 \sin nx_2 \right\} \\
&= \frac{1}{2\sin x_1} \sum_{j=1}^{m}\sum_{k=1}^{n}(q_{j-1,k}^{d} - q_{j+1,k}^{d}) \cos kx_2 \sin jx_1
\end{aligned}
$$

$$
-\frac{\sin(n+1)x_2}{4\sin x_1 \sin x_2}\left[\sum_{j=1}^{m}(\sin(j+1)x_1 - \sin(j-1)x_1) q_{j,n}^{d} - q_{mn}^{d} \sin(m+1)x_1 - q_{m+1,n}^{d} \sin mx_1\right]
$$

$$-\frac{\sin nx_2}{4\sin x_1 \sin x_2}\left[\sum_{j=1}^{m}(\sin(j+1)x_1-\sin(j-1)x_1)q^d_{j,n+1}-q^d_{m,n+1}\sin(m+1)x_1 - q^d_{m+1,n+1}\sin mx_1\right]$$

$$I_1=\frac{1}{2\sin x_1}\left[\sum_{j=1}^{m}\sum_{k=1}^{n}(\sin(j+1)x_1-\sin(j-1)x_1)q^d_{j,k}\cos kx_2-\sum_{k=1}^{n}q^d_{m,k}\sin(m+1)x_1\cos kx_2 -\sum_{k=1}^{n}q^d_{m+1,k}\sin mx_1\cos kx_2\right]-\frac{\sin(n+1)x_2}{4\sin x_1\sin x_2}\left[\sum_{j=1}^{m}(\sin(j+1)x_1-\sin(j-1)x_1)q^d_{j,n} -q^d_{mn}\sin(m+1)x_1-q^d_{m+1,n}\sin mx_1\right]-\frac{\sin nx_2}{4\sin x_1\sin x_2}\left[\sum_{j=1}^{m}(\sin(j+1)x_1-\sin(j-1)x_1)q^d_{j,n+1} -q^d_{m,n+1}\sin(m+1)x_1-q^d_{m+1,n+1}\sin mx_1\right]$$

$$=\sum_{j=1}^{m}\sum_{k=1}^{n}q^d_{j,k}\cos jx_1\cos kx_2-\frac{\sin(m+1)x_1}{2\sin x_1}\sum_{k=1}^{n}q^d_{m,k}\cos kx_2-\frac{\sin mx_1}{2\sin x_1}\sum_{k=1}^{n}q^d_{m+1,k}\cos kx_2 -\frac{\sin(n+1)x_2}{2\sin x_2}\sum_{j=1}^{m}q^d_{j,n}\cos jx_1+q^d_{m,n}\frac{\sin(m+1)x_1\sin(n+1)x_2}{4\sin x_1\sin x_2} +q^d_{m+1,n}\frac{\sin mx_1\sin(n+1)x_2}{4\sin x_1\sin x_2}-\frac{\sin nx_2}{2\sin x_2}\sum_{j=1}^{m}q^d_{j,n+1}\cos jx_1+q^d_{m,n+1}\frac{\sin(m+1)x_1\sin nx_2}{4\sin x_1\sin x_2} +q^d_{m+1,n+1}\frac{\sin mx_1\sin nx_2}{4\sin x_1\sin x_2}.$$

$$I_2=\frac{1}{4\sin x_1\sin x_2}\sum_{j=1}^{m}\sum_{k=1}^{n}(q^d_{j-1,n}-q^d_{j+1,n}-q^d_{j-1,n+2}+q^d_{j+1,n+2})\sin jx_1\sin(n+1)x_2$$

$$=\frac{n\sin(n+1)x_2}{4\sin x_1\sin x_2}\left\{\sum_{j=1}^{m}(\sin(j+1)x_1-\sin(j-1)x_1)\,q^d_{j,n}-q^d_{m,n}\sin(m+1)x_1 -q^d_{m+1,n}\sin mx_1-\sum_{j=1}^{m}(\sin(j+1)x_1-\sin(j-1)x_1)\,q^d_{j,n+2}+q^d_{m,n+2}\sin(m+1)x_1 +q^d_{m+1,n+2}\sin mx_1\right\}$$

$$= \frac{n \sin(n+1)x_2}{2\sin x_2} \sum_{j=1}^{m} \left(q^d_{j,n} - q^d_{j,n+2}\right) \cos jx_1 - \frac{\sin(m+1)x_1 \sin(n+1)x_2}{4 \sin x_1 \sin x_2} n \left(q^d_{m,n} - q^d_{m,n+2}\right)$$
$$- \frac{\sin mx_1 \sin(n+1)x_2}{4 \sin x_1 \sin x_2} n \left(q^d_{m+1,n} - q^d_{m+1,n+2}\right).$$

$$\begin{aligned} I_3 &= \frac{1}{4 \sin x_1 \sin x_2} \sum_{j=1}^{m} \sum_{k=1}^{n} (q^d_{m,k-1} - q^d_{m+2,k-1} - q^d_{m,k+1} + q^d_{m+2,k+1}) \sin(m+1)x_1 \\ &= \frac{m \sin(m+1)x_1}{2 \sin x_1} \sum_{k=1}^{n} \left(q^d_{m,k} - q^d_{m+2,k}\right) \cos kx_2 - \frac{\sin(m+1)x_1 \sin(n+1)x_2}{4 \sin x_1 \sin x_2} \\ &\qquad m(q^d_{m,n} - q^d_{m+2,n}) - \frac{\sin(m+1)x_1 \sin nx_2}{4 \sin x_1 \sin x_2} m(q^d_{m,n+1} - q^d_{m+2,n+1}). \end{aligned}$$

Finally,

$$\begin{aligned} I_4 &= \frac{1}{4 \sin x_1 \sin x_2} \sum_{j=1}^{m} \sum_{k=1}^{n} \left(q^d_{m,n} - q^d_{m+2,n} - q^d_{m,n+2} + q^d_{m+2,n+2}\right) \\ &\qquad \sin(m+1)x_1 \sin(n+1)x_2 \\ &= \frac{\sin(m+1)x_1 \sin(n+1)x_2}{4 \sin x_1 \sin x_2} mn \left(q^d_{m,n} - q^d_{m+2,n} - q^d_{m,n+2} + q^d_{m+2,n+2}\right) \end{aligned}$$

Combine all the terms $I_1, I_2, I_3, I_4$, we have

$$f^d_{mn} = S^d_{mn} - \frac{\sin(m+1)x_1}{2 \sin x_1} \sum_{k=1}^{n} q^d_{m,k} \cos kx_2 - \frac{\sin mx_1}{2 \sin x_1} \sum_{k=1}^{n} q^d_{m+1,k} \cos kx_2$$
$$- \frac{\sin(n+1)x_2}{2 \sin x_2} \sum_{j=1}^{m} q^d_{j,n} \cos jx_1 + q^d_{m,n} \frac{\sin(m+1)x_1 \sin(n+1)x_2}{4 \sin x_1 \sin x_2}$$
$$+ q^d_{m+1,n} \frac{\sin mx_1 \sin(n+1)x_2}{4 \sin x_1 \sin x_2} - \frac{\sin nx_2}{2 \sin x_2} \sum_{j=1}^{m} q^d_{j,n+1} \cos jx_1 + q^d_{m,n+1} \frac{\sin(m+1)x_1 \sin nx_2}{4 \sin x_1 \sin x_2}$$
$$+ q^d_{m+1,n+1} \frac{\sin mx_1 \sin nx_2}{4 \sin x_1 \sin x_2} - \frac{n \sin(n+1)x_2}{2 \sin x_2} \sum_{j=1}^{m} \left(q^d_{j,n} - q^d_{j,n+2}\right) \cos jx_1$$
$$+ \frac{\sin(m+1)x_1 \sin(n+1)x_2}{4 \sin x_1 \sin x_2} n(q^d_{m,n} - q^d_{m,n+2}) + \frac{\sin mx_1 \sin(n+1)x_2}{4 \sin x_1 \sin x_2} n(q^d_{m+1,n} - q^d_{m+1,n+2})$$
$$- \frac{m \sin(m+1)x_1}{2 \sin x_1} \sum_{k=1}^{n} \left(q^d_{m,k} - q^d_{m+2,k}\right) \cos kx_2 + \frac{\sin(m+1)x_1 \sin(n+1)x_2}{4 \sin x_1 \sin x_2} m(q^d_{m,n} - q^d_{m+2,n})$$

$$+\frac{\sin(m+1)x_1\sin nx_2}{4\sin x_1\sin x_2}m(q^d_{m,n+1}-q^d_{m+2,n+1})+\frac{\sin(m+1)x_1\sin(n+1)x_2}{4\sin x_1\sin x_2}mn\Big(q^d_{m,n}$$
$$-q^d_{m+2,n}-q^d_{m,n+2}+q^d_{m+2,n+2}\Big). \qquad (3.1)$$

Since $\dfrac{\sin mx_1\sin nx_2}{\sin x_1\sin x_2}$ is bounded in $(0,\pi)\times(0,\pi)$. Therefore, the terms

$$q^d_{m,n}\frac{\sin(m+1)x_1\sin(n+1)x_2}{4\sin x_1\sin x_2},\ q^d_{m+1,n}\frac{\sin mx_1\sin(n+1)x_2}{4\sin x_1\sin x_2}$$

and

$$q^d_{m,n+1}\frac{\sin(m+1)x_1\sin nx_2}{4\sin x_1\sin x_2},\ q^d_{m+1,n+1}\frac{\sin mx_1\sin nx_2}{4\sin x_1\sin x_2}$$

are of $o(1)$, as $\min(m,n)\to\infty$.

Consider the second term

$$\frac{\sin(m+1)x_1}{2\sin x_1}\sum_{k=1}^{n}q^d_{m,k}\cos kx_2$$

of the eq. (3.1) and apply abel's transformation two times on it, we get

$$= \frac{\sin(m+1)x_1}{2\sin x_1}\left[\sum_{k=1}^{n}\Delta_{02}q^d_{m,k}(k+1)F_k(x_2)+\Delta_{01}q^d_{m,n+1}(n+1)F_n(x_2)\right.$$
$$\left.+q^d_{m,n+1}D_n(x_2)\right]$$
$$\le \left|\frac{\sin(m+1)x_1}{2\sin x_1}\right|\left[\sum_{k=1}^{n}|\Delta_{02}q^d_{m,k}(k+1)F_k(x_2)|+|\Delta_{01}q^d_{m,n+1}(n+1)F_n(x_2)|\right.$$
$$\left.+|q^d_{m,n+1}D_n(x_2)|\right]$$

where $|F_n(x_2)|$ and $|D_n(x_2)|$ are bounded in $(0,\pi)$. Therefore, by the use of Lemma 2.1 all the terms of above inequality are tending to zero.

Hence,

$$\frac{\sin(m+1)x_1}{2\sin x_1}\sum_{k=1}^{n}q^d_{m,k}\cos kx_2=o(1)\ \text{as}\ \min(m,n)\to\infty.$$

Similarly, the terms

$$\frac{\sin mx_1}{2\sin x_1}\sum_{k=1}^{n}q^d_{m+1,k}\cos kx_2,\ \frac{\sin(n+1)x_2}{2\sin x_2}\sum_{j=1}^{m}q^d_{j,n}\cos jx_1 \text{ and } \frac{\sin nx_2}{2\sin x_2}\sum_{j=1}^{m}q^d_{j,n+1}\cos jx_1$$

of eq. (3.1) are of o(1), as $\min(m,n) \to \infty$.

The term
$$\frac{n\sin(n+1)x_2}{2\sin x_2}\sum_{j=1}^{m}\left(q^d_{j,n}-q^d_{j,n+2}\right)\cos jx_1$$
of the eq. (3.1) can be written as
$$= \frac{n\sin(n+1)x_2}{2\sin x_2}\sum_{j=1}^{m}(\Delta_{01}q^d_{j,n}+\Delta_{01}q^d_{j,n+1})\cos jx_1. \tag{3.2}$$
Now, performing abel's transformation two times on the eq. (3.2), we obtain
$$\begin{aligned}
&= \frac{n\sin(n+1)x_2}{2\sin x_2}\Bigg[\sum_{j=1}^{m}(\Delta_{21}q^d_{j,n}+\Delta_{21}q^d_{j,n+1})(j+1)F_j(x_1)\\
&\qquad + (\Delta_{11}q^d_{m+1,n}+\Delta_{11}q^d_{m+1,n+1})(m+1)F_m(x_1)\\
&\qquad + (\Delta_{01}q^d_{m+1,n}+\Delta_{01}q^d_{m+1,n+1})D_m(x_1)\Bigg]\\
&\le C\Bigg[\sum_{j=1}^{m}\sum_{k=n}^{\infty}|(\Delta_{22}q^d_{j,k}+\Delta_{22}q^d_{j,k+1})(j+1)(k+1)|\\
&\qquad + |(m+1)n(\Delta_{11}q^d_{m+1,n}+\Delta_{11}q^d_{m+1,n+1})| + |n(\Delta_{01}q^d_{m+1,n}+\Delta_{01}q^d_{m+1,n+1})|\Bigg]\\
&= o(1) \text{ as } \min(m,n)\to\infty \text{ (by given hypothesis).}
\end{aligned}$$
Similarly,
$$\frac{m\sin(m+1)x_1}{2\sin x_1}\sum_{k=1}^{n}(q^d_{m,k}-q^d_{m+2,k})\cos kx_2 = o(1),\ \ \min(m,n)\to\infty.$$
The term
$$\frac{\sin(m+1)x_1\sin(n+1)x_2}{4\sin x_1\sin x_2}n(q^d_{m,n}-q^d_{m,n+2})$$
of eq. (3.1) can be written as
$$\begin{aligned}
&= \frac{\sin(m+1)x_1\sin(n+1)x_2}{4\sin x_1\sin x_2}n(\Delta_{01}q^d_{m,n}+\Delta_{01}q^d_{m,n+1})\\
&= o(1),\ \ \min(m,n)\to\infty \text{ (by using Lemma 2.1).}
\end{aligned}$$

Similarly, the terms

$$\frac{\sin mx_1 \sin(n+1)x_2}{4\sin x_1 \sin x_2} n(q^d_{m+1,n} - q^d_{m+1,n+2}), \frac{\sin(m+1)x_1 \sin(n+1)x_2}{4\sin x_1 \sin x_2} m(q^d_{m,n} - q^d_{m+2,n})$$

and

$\frac{\sin(m+1)x_1 \sin nx_2}{4\sin x_1 \sin x_2} m(q^d_{m,n+1} - q^d_{m+2,n+1})$ of eq. (3.1) are of o(1), as $\min(m,n) \to \infty$.

The last term

$$\frac{\sin(m+1)x_1 \sin(n+1)x_2}{4\sin x_1 \sin x_2} mn \left(q^d_{m,n} - q^d_{m+2,n} - q^d_{m,n+2} + q^d_{m+2,n+2}\right)$$

of the eq. (3.1) can be written as

$$\begin{aligned} &= \frac{\sin(m+1)x_1 \sin(n+1)x_2}{4\sin x_1 \sin x_2} mn \left(\Delta_{11} q^d_{m,n} + \Delta_{11} q^d_{m,n+1} + \Delta_{11} q^d_{m+1,n} + \Delta_{11} q^d_{m+1,n+1}\right) \\ &= o(1) \ as \ \min(m,n) \to \infty \quad \text{(by using Lemma 2.1).} \end{aligned}$$

Further, consider

$$\begin{aligned} S^d_{mn}(x_1,x_2) &= \sum_{j=1}^{m} \sum_{k=1}^{n} q^d_{jk} \cos jx_1 \cos kx_2 \\ &= \sum_{j=0}^{m} \sum_{k=0}^{n} q^d_{jk} \cos jx_1 \cos kx_2, \quad (\because q^d_{j,0} = q^d_{0,k} = 0 \ \ \forall j,k) \end{aligned}$$

use of double summation by parts yields

$$\begin{aligned} S^d_{mn} &= \sum_{j=0}^{m} \sum_{k=0}^{n} \Delta_{11} q^d_{jk} D_j(x_1) D_k(x_2) + \sum_{k=0}^{n} \Delta_{01} q^d_{m+1,k} D_k(x_2) D_m(x_1) \\ &\quad + \sum_{j=0}^{m} \Delta_{10} q^d_{j,n+1} D_j(x_1) D_n(x_2) + q^d_{m+1,n+1} D_m(x_1) D_n(x_2) \end{aligned}$$

Again on applying double summation by parts, we get

$$\begin{aligned}
S^d_{mn} &= \sum_{j=0}^{m}\sum_{k=0}^{n}\Delta_{22}q^d_{jk}(j+1)(k+1)F_j(x_1)F_k(x_2) + \sum_{k=0}^{n}\Delta_{12}q^d_{m+1,k}(k+1)(m+1)F_k(x_2)F_m(x_1)\\
&+\sum_{j=0}^{m}\Delta_{21}q^d_{j,n+1}(j+1)(n+1)F_j(x_1)F_n(x_2) + \Delta_{11}q^d_{m+1,n+1}(m+1)(n+1)F_m(x_1)F_n(x_2)\\
&+\sum_{k=0}^{n}\Delta_{02}q^d_{m+1,k}(k+1)F_k(x_2)D_m(x_1) + \sum_{j=0}^{m}\Delta_{20}q^d_{j,n+1}(j+1)F_j(x_1)D_n(x_2)\\
&+\Delta_{01}q^d_{m+1,n+1}(n+1)F_n(x_2)D_m(x_1) + \Delta_{01}q^d_{m+1,n+1}(m+1)F_m(x_1)D_n(x_2)\\
&\qquad +q^d_{m+1,n+1}D_m(x_1)D_n(x_2)\\
&\le \sum_{j=0}^{m}\sum_{k=0}^{n}|\Delta_{22}q^d_{jk}(j+1)(k+1)F_j(x_1)F_k(x_2)| + \sum_{j=m+1}^{\infty}\sum_{k=0}^{n}|\Delta_{22}q^d_{j,k}(j+1)(k+1)F_j(x_1)F_k(x_2)|\\
&+\sum_{j=0}^{m}\sum_{k=n+1}^{\infty}|\Delta_{22}q^d_{j,k}(j+1)(k+1)F_j(x_1)F_k(x_2)| + |\Delta_{11}q^d_{m+1,n+1}(m+1)(n+1)F_m(x_1)F_n(x_2)|\\
&+\sum_{k=0}^{n}|\Delta_{02}q^d_{m+1,k}(k+1)F_k(x_2)D_m(x_1)| + \sum_{j=0}^{m}|\Delta_{20}q^d_{j,n+1}(j+1)F_j(x_1)D_n(x_2)|\\
&+|\Delta_{01}q^d_{m+1,n+1}(n+1)F_n(x_2)D_m(x_1)| + |\Delta_{01}q^d_{m+1,n+1}(m+1)F_m(x_1)D_n(x_2)|\\
&\qquad +|q^d_{m+1,n+1}D_m(x_1)D_n(x_2)|
\end{aligned}$$

For all $0 < x_1, x_2 < \pi$, $S^d_{mn}(x_1,x_2)$ converges to $f^d(x_1,x_2)$ as $\min(m,n)\to\infty$ (by the use of given hypothesis and Lemma 2.1).
Hence,

$$f^d(x_1,x_2) = \lim_{\min(m,n)\to\infty} S^d_{mn}(x_1,x_2) = \lim_{\min(m,n)\to\infty} f^d_{mn}(x_1,x_2) \ \text{ exists.}$$

To prove (ii), consider

$$\int_0^{\pi}\int_0^{\pi}|f^d(x_1,x_2)|dx_1dx_2 = \int_0^{\pi}\int_0^{\pi}\left|\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}q^d_{j,k}\cos jx_1\cos kx_2\right|dx_1dx_2$$

Performing double summation by parts two times, we get

$$\int_0^{\pi}\int_0^{\pi}|f^d(x_1,x_2)|dx_1dx_2 = \int_0^{\pi}\int_0^{\pi}\left|\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta_{22}q^d_{j,k}(j+1)(k+1)F_j(x_1)F_k(x_2)\right|dx_1dx_2$$

Since $\frac{1}{\pi}\int\limits_{-\pi}^{\pi}|F_j(x_1)|dx_1 = 1$, therefore

$$\begin{aligned}\int\limits_0^{\pi}\int\limits_0^{\pi}|f^d(x_1,x_2)|dx_1dx_2 &= O\left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\Delta_{22}q^d_{j,k}|(j+1)(k+1)\right)\\ &< \infty \text{ (by the given hypothesis).}\end{aligned}$$

Now, consider

$$\begin{aligned}||f^d - f^d_{mn}|| = \Bigg\| &\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty} q^d_{j,k}\cos jx_1\cos kx_2 + \frac{\sin(m+1)x_1}{2\sin x_1}\sum_{k=1}^{n} q^d_{m,k}\cos kx_2\\
&+ \frac{\sin mx_1}{2\sin x_1}\sum_{k=1}^{n} q^d_{m+1,k}\cos kx_2 + \frac{\sin(n+1)x_2}{2\sin x_2}\sum_{j=1}^{m} q^d_{j,n}\cos jx_1\\
&- q^d_{m,n}\frac{\sin(m+1)x_1\sin(n+1)x_2}{4\sin x_1\sin x_2} - q^d_{m+1,n}\frac{\sin mx_1\sin(n+1)x_2}{4\sin x_1\sin x_2}\\
&+ \frac{\sin nx_2}{2\sin x_2}\sum_{j=1}^{m} q^d_{j,n+1}\cos jx_1 - q^d_{m,n+1}\frac{\sin(m+1)x_1\sin nx_2}{4\sin x_1\sin x_2}\\
&- q^d_{m+1,n+1}\frac{\sin mx_1\sin nx_2}{4\sin x_1\sin x_2} + \frac{n\sin(n+1)x_2}{2\sin x_2}\sum_{j=1}^{m}\left(q^d_{j,n} - q^d_{j,n+2}\right)\cos jx_1\\
&- \frac{\sin(m+1)x_1\sin(n+1)x_2}{4\sin x_1\sin x_2}n(q^d_{m,n} - q^d_{m,n+2}) - \frac{\sin mx_1\sin(n+1)x_2}{4\sin x_1\sin x_2}n(q^d_{m+1,n} - q^d_{m+1,n+2})\\
&+ \frac{m\sin(m+1)x_1}{2\sin x_1}\sum_{k=1}^{n}\left(q^d_{m,k} - q^d_{m+2,k}\right)\cos kx_2\\
&- \frac{\sin(m+1)x_1\sin(n+1)x_2}{4\sin x_1\sin x_2}m(q^d_{m,n} - q^d_{m+2,n}) - \frac{\sin(m+1)x_1\sin nx_2}{4\sin x_1\sin x_2}m(q^d_{m,n+1} - q^d_{m+2,n+1})\\
&- \frac{\sin(m+1)x_1\sin(n+1)x_2}{4\sin x_1\sin x_2}mn\left(q^d_{m,n} - q^d_{m+2,n} - q^d_{m,n+2} + q^d_{m+2,n+2}\right)\Bigg\|\end{aligned}$$

$$\begin{aligned}||f^d - f^d_{mn}|| \le &\int\limits_0^{\pi}\int\limits_0^{\pi}\left|\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty} q^d_{j,k}\cos jx_1\cos kx_2\right|dx_1dx_2 + \int\limits_0^{\pi}\int\limits_0^{\pi}\left|\frac{\sin(m+1)x_1}{2\sin x_1}\sum_{k=1}^{n} q^d_{m,k}\cos kx_2\right|dx_1dx_2\\
&+ \int\limits_0^{\pi}\int\limits_0^{\pi}\left|\frac{\sin mx_1}{2\sin x_1}\sum_{k=1}^{n} q^d_{m+1,k}\cos kx_2\right|dx_1dx_2 + \int\limits_0^{\pi}\int\limits_0^{\pi}\left|\frac{\sin(n+1)x_2}{2\sin x_2}\sum_{j=1}^{m} q^d_{j,n}\cos jx_1\right|dx_1dx_2\\
&+ \int\limits_0^{\pi}\int\limits_0^{\pi}\left|q^d_{m,n}\frac{\sin(m+1)x_1\sin(n+1)x_2}{4\sin x_1\sin x_2}\right|dx_1dx_2 + \int\limits_0^{\pi}\int\limits_0^{\pi}\left|q^d_{m+1,n}\frac{\sin mx_1\sin(n+1)x_2}{4\sin x_1\sin x_2}\right|dx_1dx_2\end{aligned}$$

$$+\int_0^\pi\int_0^\pi\left|\frac{\sin nx_2}{2\sin x_2}\sum_{j=1}^{m}q^d_{j,n+1}\cos jx_1\right|dx_1dx_2+\int_0^\pi\int_0^\pi\left|q^d_{m,n+1}\frac{\sin(m+1)x_1\sin nx_2}{4\sin x_1\sin x_2}\right|dx_1dx_2$$

$$+\int_0^\pi\int_0^\pi\left|q^d_{m+1,n+1}\frac{\sin mx_1\sin nx_2}{4\sin x_1\sin x_2}\right|dx_1dx_2$$

$$+\int_0^\pi\int_0^\pi\left|\frac{n\sin(n+1)x_2}{2\sin x_2}\sum_{j=1}^{m}\left(q^d_{j,n}-q^d_{j,n+2}\right)\cos jx_1\right|dx_1dx_2$$

$$+\int_0^\pi\int_0^\pi\left|\frac{\sin(m+1)x_1\sin(n+1)x_2}{4\sin x_1\sin x_2}n(q^d_{m,n}-q^d_{m,n+2})\right|dx_1dx_2$$

$$+\int_0^\pi\int_0^\pi\left|\frac{\sin mx_1\sin(n+1)x_2}{4\sin x_1\sin x_2}n(q^d_{m+1,n}-q^d_{m+1,n+2})\right|dx_1dx_2$$

$$+\int_0^\pi\int_0^\pi\left|\frac{m\sin(m+1)x_1}{2\sin x_1}\sum_{k=1}^{n}\left(q^d_{m,k}-q^d_{m+2,k}\right)\cos kx_2\right|dx_1dx_2$$

$$+\int_0^\pi\int_0^\pi\left|\frac{\sin(m+1)x_1\sin(n+1)x_2}{4\sin x_1\sin x_2}m(q^d_{m,n}-q^d_{m+2,n})\right|dx_1dx_2$$

$$+\int_0^\pi\int_0^\pi\left|\frac{\sin(m+1)x_1\sin nx_2}{4\sin x_1\sin x_2}m(q^d_{m,n+1}-q^d_{m+2,n+1})\right|dx_1dx_2$$

$$+\int_0^\pi\int_0^\pi\left|\frac{\sin(m+1)x_1\sin(n+1)x_2}{4\sin x_1\sin x_2}mn\left(q^d_{m,n}-q^d_{m+2,n}-q^d_{m,n+2}+q^d_{m+2,n+2}\right)\right|dx_1dx_2$$

$$=\int\int_1+\int\int_2+\int\int_3+...+\int\int_{15}+\int\int_{16} \tag{3.3}$$

The integral $\int\int_5$ from eq. (3.3) is

$$\int_0^\pi\int_0^\pi\left|q^d_{m,n}\frac{\sin(m+1)x_1\sin(n+1)x_2}{4\sin x_1\sin x_2}\right|dx_1dx_2=|q^d_{m,n}\ln m\ln n|$$

$$=o(1)\text{ as }\min(m,n)\to\infty.$$

Similarly, the integrals $\int\int_6, \int\int_8$ and $\int\int_9$ from eq. (3.3) are

$$\int_0^\pi\int_0^\pi\left|q^d_{m+1,n}\frac{\sin mx_1\sin(n+1)x_2}{4\sin x_1\sin x_2}\right|dx_1dx_2=o(1)\text{ as }\min(m,n)\to\infty,$$

$$\int_0^\pi\int_0^\pi \left| q^d_{m,n+1}\frac{\sin(m+1)x_1\sin nx_2}{4\sin x_1\sin x_2}\right| dx_1dx_2 = o(1) \text{ as } \min(m,n)\to\infty,$$

$$\int_0^\pi\int_0^\pi \left| q^d_{m+1,n+1}\frac{\sin mx_1\sin nx_2}{4\sin x_1\sin x_2}\right| dx_1dx_2 = o(1) \text{ as } \min(m,n)\to\infty.$$

The integral $\int\int_{11}$ of eq. (3.3) can be written as

$$\begin{aligned}\int\int_{11} &= |n(\Delta_{01}q^d_{m,n}+\Delta_{01}q^d_{m,n+1})\ln m\ln n| \\ &= |n\Delta_{01}q^d_{m,n}\ln m\ln n| + |n\Delta_{01}q^d_{m,n+1}\ln m\ln n| \\ &= o(1), \min(m,n)\to\infty.\end{aligned}$$

Hence,

$$\int_0^\pi\int_0^\pi \left| n(q^d_{m,n}-q^d_{m,n+2})\frac{\sin(m+1)x_1\sin(n+1)x_2}{4\sin x_1\sin x_2}\right| dx_1dx_2 = o(1), \min(m,n)\to\infty$$

Similarly, the integrals $\int\int_{12}, \int\int_{14}$ and $\int\int_{15}$ of eq. (3.3) are

$$\int_0^\pi\int_0^\pi \left| n(q^d_{m+1,n}-q^d_{m+1,n+2})\frac{\sin mx_1\sin(n+1)x_2}{4\sin x_1\sin x_2}\right| dx_1dx_2 = o(1), \min(m,n)\to\infty$$

$$\int_0^\pi\int_0^\pi \left| m(q^d_{m,n}-q^d_{m+2,n})\frac{\sin(m+1)x_1\sin(n+1)x_2}{4\sin x_1\sin x_2}\right| dx_1dx_2 = o(1), \min(m,n)\to\infty$$

$$\int_0^\pi\int_0^\pi \left| m(q^d_{m,n+1}-q^d_{m+2,n+1})\frac{\sin(m+1)x_1\sin nx_2}{4\sin x_1\sin x_2}\right| dx_1dx_2 = o(1), \min(m,n)\to\infty$$

The last integral $\int\int_{16}$ of the eq. (3.3) is

$$\int_0^\pi\int_0^\pi \left| mn(q^d_{m,n}-q^d_{m+2,n}-q^d_{m,n+2}+q^d_{m+2,n+2})\frac{\sin(m+1)x_1\sin(n+1)x_2}{4\sin x_1\sin x_2}\right| dx_1dx_2$$

$$= mn\left|\Delta_{11}q^d_{m,n}+\Delta_{11}q^d_{m,n+1}+\Delta_{11}q^d_{m+1,n}+\Delta_{11}q^d_{m+1,n+1}\right|\ln m\ln n$$

$$= o(1), \ \min(m,n)\to\infty.$$

Now, consider the integral $\int\int_2$ from eq. (3.3) and apply summation by parts on it, we get

$$\begin{aligned}\int\int_2 &= \int_0^\pi\int_0^\pi \left|\frac{\sin(m+1)x_1}{2\sin x_1}\left[\sum_{k=1}^{n}\Delta_{01}q_{m,k}^d D_k(x_2) + q_{m,n+1}^d D_n(x_2)\right]\right| dx_1dx_2\\ &= \int_0^\pi\int_0^\pi \left|\frac{\sin(m+1)x_1}{2\sin x_1}\left[\sum_{k=1}^{n}\Delta_{02}q_{m,k}^d (k+1)F_k(x_2) + \Delta_{01}q_{m,n+1}^d(n+1)F_n(x_2)\right.\right.\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.\left. + q_{m,n+1}^d D_n(x_2)\right]\right| dx_1dx_2\\ &\leq \pi\left[\sum_{k=1}^{n}|\Delta_{02}q_{m,k}^d(k+1)\ln m| + |\Delta_{01}q_{m,n+1}^d(n+1)\ln m|\right] + |q_{m,n+1}^d \ln m \ln n|\end{aligned}$$

By using Lemma 2.1, all the terms of above inequality are of $o(1)$ as $\min(m,n)\to\infty$. Hence

$$\int_0^\pi\int_0^\pi \left|\frac{\sin(m+1)x_1}{2\sin x_1}\sum_{k=1}^{n} q_{m,k}^d\cos kx_2\right| dx_1dx_2 = o(1), \ \ \min(m,n)\to\infty.$$

Similarly, the integrals $\int\int_3, \int\int_4$ and $\int\int_7$ from eq. (3.3) are

$$\begin{aligned}\int_0^\pi\int_0^\pi \left|\frac{\sin mx_1}{2\sin x_1}\sum_{k=1}^{n} q_{m+1,k}^d\cos kx_2\right| dx_1dx_2 &= o(1), \ \ \min(m,n)\to\infty,\\ \int_0^\pi\int_0^\pi \left|\frac{\sin(n+1)x_2}{2\sin x_2}\sum_{j=1}^{m} q_{j,n}^d\cos jx_1\right| dx_1dx_2 &= o(1), \ \ \min(m,n)\to\infty,\\ \int_0^\pi\int_0^\pi \left|\frac{\sin nx_2}{2\sin x_2}\sum_{j=1}^{m} q_{j,n+1}^d\cos jx_1\right| dx_1dx_2 &= o(1), \ \ \min(m,n)\to\infty.\end{aligned}$$

The integral $\int\int_{10}$ of eq. (3.3) can be written as

$$\begin{aligned}\int\int_{10} &= n\ln n\int_0^\pi\left|\sum_{j=1}^{m}(q_{j,n}^d - q_{j,n+2}^d)\cos jx_1\right| dx_1\\ &\leq \ln n\int_0^\pi\left|\sum_{j=1}^{\infty}\sum_{k=n}^{\infty}k(\Delta_{01}q_{j,k}^d - \Delta_{01}q_{j,k+2}^d)\cos jx_1\right| dx_1\end{aligned}$$

Apply summation by parts, we get

$$\begin{aligned}\int\int_{10} &= \ln n \int_0^{\pi} \left| \sum_{j=1}^{\infty} \sum_{k=n}^{\infty} k(\Delta_{11} q_{j,k}^d - \Delta_{11} q_{j,k+2}^d) D_j(x_1) \right| dx_1 \\ &= \ln n \int_0^{\pi} \left| \sum_{j=1}^{\infty} \sum_{k=n}^{\infty} k(\Delta_{12} q_{j,k}^d + \Delta_{12} q_{j,k+1}^d) D_j(x_1) \right| dx_1\end{aligned}$$

Again on applying summation by parts, we have

$$\begin{aligned}\int\int_{10} &= \ln n \int_0^{\pi} \left| \sum_{j=1}^{\infty} \sum_{k=n}^{\infty} k(\Delta_{22} q_{j,k}^d + \Delta_{22} q_{j,k+1}^d)(j+1) F_j(x_1) \right| dx_1 \\ &\leq \pi \ln n \sum_{j=1}^{\infty} \sum_{k=n}^{\infty} (j+1)(k+1) \left| \Delta_{22} q_{j,k}^d + \Delta_{22} q_{j,k+1}^d \right| \\ &\leq \pi \left[ \sum_{j=1}^{\infty} \sum_{k=n}^{\infty} (j+1)(k+1) |\Delta_{22} q_{j,k}^d| \ln(k+2) \right. \\ &+ \left. \sum_{j=1}^{\infty} \sum_{k=n}^{\infty} (j+1)(k+1) |\Delta_{22} q_{j,k+1}^d| \ln(k+2) \right] \\ &= o(1), \ \ \min(m,n) \to \infty.\end{aligned}$$

Hence,

$$\int_0^{\pi} \int_0^{\pi} \left| n \sum_{j=1}^{m} (q_{j,n}^d - q_{j,n+2}^d) \cos jx_1 \frac{\sin(n+1)x_2}{2\sin x_2} \right| dx_1 dx_2 = o(1), \min(m,n) \to \infty.$$

Similarly, the integral $\int\int_{13}$ of eq. (3.3) is

$$\int_0^{\pi} \int_0^{\pi} \left| m \sum_{k=1}^{n} (q_{m,k}^d - q_{m+2,k}^d) \cos kx_2 \frac{\sin(m+1)x_1}{2\sin x_1} \right| dx_1 dx_2 \quad = \quad o(1), \min(m,n) \to \infty.$$

The remaining integral $\int\int_1$ of eq. (3.3) is

$$\int\int_1 = \int_0^{\pi} \int_0^{\pi} \left| \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} q_{j,k}^d \cos jx_1 \cos kx_2 \right| dx_1 dx_2$$

Apply double summation by parts, we get

$$\int\int_1 = \int_0^\pi\int_0^\pi \left|\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}\Delta_{11}q_{j,k}^d D_j(x_1)D_k(x_2) - \sum_{k=n+1}^{\infty}\Delta_{01}q_{m+1,k}^d D_m(x_1)D_k(x_2)\right.$$
$$\left. - \sum_{j=m+1}^{\infty}\Delta_{10}q_{j,n+1}^d D_j(x_1)D_n(x_2) + q_{m+1,n+1}^d D_m(x_1)D_n(x_2)\right| dx_1dx_2$$

Again on performing double summation by parts, we have

$$\begin{aligned}\int\int_1 \leq\ & \int_0^\pi\int_0^\pi \left|\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}\Delta_{22}q_{j,k}^d (j+1)(k+1)F_j(x_1)F_k(x_2)\right| dx_1dx_2 \\ & + \int_0^\pi\int_0^\pi \left|\sum_{k=n+1}^{\infty}\Delta_{12}q_{m+1,k}^d (m+1)(k+1)F_m(x_1)F_k(x_2)\right| dx_1dx_2 \\ & + \int_0^\pi\int_0^\pi \left|\sum_{j=m+1}^{\infty}\Delta_{21}q_{j,n+1}^d (j+1)(n+1)F_j(x_1)F_n(x_2)\right| dx_1dx_2 \\ & + \int_0^\pi\int_0^\pi \left|\Delta_{11}q_{m+1,n+1}^d (m+1)(n+1)F_m(x_1)F_n(x_2)\right| dx_1dx_2 \\ & + \int_0^\pi\int_0^\pi \left|\sum_{k=n+1}^{\infty}\Delta_{02}q_{m+1,k}^d (k+1)D_m(x_1)F_k(x_2)\right| dx_1dx_2 \\ & + \int_0^\pi\int_0^\pi \left|\sum_{j=m+1}^{\infty}\Delta_{20}q_{j,n+1}^d (j+1)F_j(x_1)D_n(x_2)\right| dx_1dx_2 \\ & + \int_0^\pi\int_0^\pi \left|\Delta_{01}q_{m+1,n+1}^d (n+1)D_m(x_1)F_n(x_2)\right| dx_1dx_2 \\ & + \int_0^\pi\int_0^\pi \left|\Delta_{10}q_{m+1,n+1}^d (m+1)F_m(x_1)D_n(x_2)\right| dx_1dx_2 \\ & + \int_0^\pi\int_0^\pi \left|q_{m+1,n+1}^d D_m(x_1)D_n(x_2)\right| dx_1dx_2\end{aligned}$$

$$\begin{aligned}
\int\int_1 \;\leq\;& C\Big[\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}|\Delta_{22}q^d_{j,k}|(j+1)(k+1)\\
&+\left|\Delta_{11}q^d_{m+1,n+1}(m+1)(n+1)\right|+\sum_{k=n+1}^{\infty}\left|\Delta_{02}q^d_{m+1,k}(k+1)\ln m\right|\\
&+\sum_{j=m+1}^{\infty}\left|\Delta_{20}q^d_{j,n+1}(j+1)\ln n\right|+|\Delta_{01}q^d_{m+1,n+1}(n+1)\ln m|\\
&+|\Delta_{10}q^d_{m+1,n+1}(m+1)\ln n|\Big]+\left|q^d_{m+1,n+1}\ln m\ln n\right|\\
=\;& o(1) \text{ as } \min(m,n)\to\infty \text{ (by given hypothesis)}
\end{aligned}$$

Combining all the terms, we get

$$||f^d-f^d_{mn}||=o(1) \text{ as } \min(m,n)\to\infty$$

**Remark 3.2.** As we know that a double trigonometric series converges in $L^1$-norm to a function $f^d\in L^1(Q)$ then it is a Fourier series of function $f^d$.Thus by Theorem 3.1, we get (1.1) is a Fourier cosine series of $f^d$.

**Corollary 3.3** *If a double sequence $\{q^d_{jk}\}$ belongs to the class $\mathcal{S}_{jk}$, then* $||S^d_{mn}-f^d||\to 0$ *as* $\min(m,n)\to\infty$.

**Proof.**

$$\begin{aligned}
&||f^d-S^d_{mn}||=||f^d-f^d_{mn}+f^d_{mn}-S^d_{mn}||\leq||f^d-f^d_{mn}||+||f^d_{mn}-S^d_{mn}||\\
=\;&||f^d-f^d_{mn}||+\Bigg|\Bigg|-\frac{\sin(m+1)x_1}{2\sin x_1}\sum_{k=1}^{n}q^d_{m,k}\cos kx_2-\frac{\sin mx_1}{2\sin x_1}\sum_{k=1}^{n}q^d_{m+1,k}\cos kx_2\\
&-\frac{\sin(n+1)x_2}{2\sin x_2}\sum_{j=1}^{m}q^d_{j,n}\cos jx_1+q^d_{m,n}\frac{\sin(m+1)x_1\sin(n+1)x_2}{4\sin x_1\sin x_2}\\
&+q^d_{m+1,n}\frac{\sin mx_1\sin(n+1)x_2}{4\sin x_1\sin x_2}-\frac{\sin nx_2}{2\sin x_2}\sum_{j=1}^{m}q^d_{j,n+1}\cos jx_1\\
&+q^d_{m,n+1}\frac{\sin(m+1)x_1\sin nx_2}{4\sin x_1\sin x_2}+q^d_{m+1,n+1}\frac{\sin mx_1\sin nx_2}{4\sin x_1\sin x_2}\\
&-\frac{n\sin(n+1)x_2}{2\sin x_2}\sum_{j=1}^{m}\left(q^d_{j,n}-q^d_{j,n+2}\right)\cos jx_1\\
&+\frac{\sin(m+1)x_1\sin(n+1)x_2}{4\sin x_1\sin x_2}n(q^d_{m,n}-q^d_{m,n+2})
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\sin mx_1 \sin(n+1)x_2}{4\sin x_1 \sin x_2} n(q^d_{m+1,n} - q^d_{m+1,n+2}) \\
&- \frac{m\sin(m+1)x_1}{2\sin x_1} \sum_{k=1}^{n} \left(q^d_{m,k} - q^d_{m+2,k}\right) \cos kx_2 \\
&+ \frac{\sin(m+1)x_1 \sin(n+1)x_2}{4\sin x_1 \sin x_2} m(q^d_{m,n} - q^d_{m+2,n}) \\
&+ \frac{\sin(m+1)x_1 \sin nx_2}{4\sin x_1 \sin x_2} m(q^d_{m,n+1} - q^d_{m+2,n+1}) \\
&+ \ \frac{\sin(m+1)x_1 \sin(n+1)x_2}{4\sin x_1 \sin x_2} mn\left(q^d_{m,n} - q^d_{m+2,n} - q^d_{m,n+2} + q^d_{m+2,n+2}\right)\Bigg\|.
\end{aligned}
$$

Using Theorem 3.1, all the terms on right hand side of above inequality are of $o(1)$ as $\min(m,n) \to \infty$. Thus, the conclusion of the corollary follows.

## References


[1] J. Kaur and S.S. Bhatia, Integrability and $L^1$-convergence of double cosine trigonometric series, Analysis in Theory and Applications, 27(1)(2011), 32-39.

[2] K. Kaur, S.S. Bhatia and B. Ram, Double trigonometric series with coefficients of bounded variation of higher order, Tamkang J. Math., 35(4)(2004), 267-280.

[3] X.Z. Krasniqi, A note on $L^1$-convergence of the sine and cosine trigonometric series with semi-convex coefficients, Int. J. Open Problems Comput. Sci. Math., 2(2)(2009), 231-239.

[4] F. Móricz, Convergence and integrability of double trigonometric series with coefficients of bounded variation, Proc. Amer. Math. Soc., 3(1988), 633-640.

[5] F. Móricz, On the integrability and $L^1$-convergence of sine series, Studia Math., 92(1989), 187-200.

[6] F. Móricz, Integrability of double trigonometric series with special coefficients Anal. Math., 16(1990), 39-56.

[7] F. Móricz, On the integrability and $L^1$ convergence of double trigonometric series, Studia Math., 98(3) (1991), 203-225.